between the trailing vortices is very weak and the instability of each vortex can be regarded as the symmetrical Crow instability caused by mutual induction of the vortex and its mirror image. The mean lateral displacement of vortices is not important.

In the middle region of H, the vortex is inclined to the axis ox at the angles θ_y and θ_z, which may be estimated as

$$\theta_y \sim -\frac{\bar{\Gamma}}{2\pi A_R}\frac{4H^2}{1+4H^2}, \qquad \theta_z \sim \frac{\bar{\Gamma}}{4\pi H A_R}\frac{1}{1+4H^2}$$

where the circulation $\bar{\Gamma}$ is nondimensionalized with respect to the wing chord c. It can be shown that the angles θ_y and θ_z can be neglected throughout the interval $0.1 \leq H \leq 1.5$ for typical wings of aircraft. The underlying assumptions related to the geometry of the unperturbed vortices also are acceptable in the middle region. Consequently the theory presented is correct in inviscid fluid, at least qualitatively.

Conclusion

As follows from the presented linear analysis in inviscid fluid, the trailing vortices have undergone a growing sinusoidal instability in close vicinity to the ground and have come into contact with it. However, in reality the approaching vortices generate the boundary layer along the ground that separates and causes the primary vortices to follow a complicated trajectory. Our next paper will be devoted to the nonlinear instability of the trailing vortices in a viscous flow near the ground.

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Scaling Laws of Cylindrical Shells Under Lateral Pressure

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Introduction

BECAUSE of the complexity of the laminated composite structures and lack of complete design-based information, any new design must be extensively evaluated by experiments until it achieves the necessary reliability and safety. However, the experimental evaluation of these structures is costly and time consuming. More importantly, it is impractical to test large structures. Consequently, it is extremely useful if the behavior of a full-scale structure can be predicted from the behavior of a similar small-scale model.

Understanding the relationship between model and prototype behavior is essential in designing scaled-down models. To better understand the applicability of structural similitude in designing laminated composite structures, an analytical investigation was undertaken to assess the feasibility of its utility. Such a study is important because it provides the necessary scaling laws and scale factors that affect the accuracy of the predicted response.

Similitude theory has a wide application in solid, fluid, electrical, and thermal systems.[1] Structural similitude theory[2] is proven to be a very useful tool. In this theory, similarity is established in the solutions of the governing equations of the scaled-down model (scale model) and the full-scale structure (prototype).

The main objective of this study is to demonstrate the applicability of similitude theory in designing scaled-down models for predicting the buckling behavior of laminated cylindrical shells subjected to lateral pressure. Based on the resulting scaling laws, a new prediction equation is developed to establish the behavior of prototype both for complete and partial similarities.

Buckling Equations

Donnell-type equations governing buckling of laminated shells are commonly expressed in terms of variations of in-plane force and moment resultants, which can be subsequently expressed in terms of variations of displacements during buckling. The solution to the governing differential equations for laminated cross-ply shells with simply supported boundary conditions is straightforward. The result is a simple closed-form buckling criterion that is applicable for stress resultant $\bar{N}_{yy}$ due to lateral pressure (for details, see Ref. 3) as follows:

$$T_{33} + \frac{(2T_{12}T_{13}T_{23} - T_{11}T_{23}^2 - T_{22}T_{13}^2)}{(T_{11}T_{22} - T_{12}^2)} = -\bar{N}_{yy}\xi^2 \qquad (1)$$

where

$$\eta = \bar{m}\pi/L, \qquad \xi = \bar{n}/R$$

$$T_{11} = A_{11}\eta^2 + A_{66}\xi^2, \qquad T_{22} = A_{22}\xi^2 + A_{66}\eta^2$$

$$T_{12} = (A_{12} + A_{66})\xi\eta, \qquad T_{23} = A_{22}R^{-1}\xi$$

$$T_{13} = A_{12}R^{-1}\eta, \qquad T_{33} = D_{11}\eta^4 + 2\bar{D}_{12}\xi^2\eta^2 + D_{22}\xi^4 + A_{22}R^{-2} \qquad (2)$$

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Equation (1) predicts eigenvalues, the lowest of which corresponds to the buckling load. The integers $\bar{m}$ and $\bar{n}$ are the number of axial and circumferential half-waves and full waves, respectively. A minimization with respect to integer values of $\bar{m}$ and $\bar{n}$ is performed to find the lowest eigenvalue.

Scaling Laws

Assuming Eq. (1) represents the response of both the prototype and its models and applying similitude theory to the preceding equation establishes the following scaling laws for symmetric, cross-ply, laminated cylinders subjected to lateral pressure $\bar{N}_{yy}$ (see Refs. 2 and 4):

$$\lambda_{K_{yy}} = \left(\lambda_\eta^4 \big/ \lambda_\xi^2\right)\lambda_L^2 \tag{3}$$

$$\lambda_{K_{yy}} = \left(\lambda_{\bar{D}_{12}} \big/ \lambda_{D_{11}}\right)\lambda_\eta^2\lambda_L^2 \tag{4}$$

$$\lambda_{K_{yy}} = \frac{\lambda_{A_{22}}}{\lambda_{D_{11}}}\frac{\lambda_L^2}{\lambda_R^2\lambda_\xi^2} \tag{5}$$

$$\lambda_{K_{yy}} = \frac{\lambda_{\bar{A}_{12}}^2}{\lambda_{D_{11}}\lambda_{A_{11}}}\frac{\lambda_L^2}{\lambda_R^2\lambda_\xi^2} \tag{6}$$

$$\lambda_{K_{yy}} = \left(\lambda_{D_{22}} \big/ \lambda_{D_{11}}\right)\lambda_\xi^2\lambda_L^2 \tag{7}$$

$$\lambda_{K_{yy}} = \frac{\lambda_{\bar{N}_{yy}}}{\lambda_{D_{11}}}\lambda_L^2 \tag{8}$$

where

$$K_{yy} = -\frac{\bar{N}_{yy}L^2}{\pi^2 D_{11}}, \qquad \bar{A}_{12} = A_{12} + A_{66}, \qquad \bar{D}_{12} = D_{12} + 2D_{66}$$

and $\lambda_{x_i} = X_{ip}/X_{im}$ is the scale factor of parameter X_i (p and m are prototype and model, respectively).

The preceding equations are necessary to predict the behavior of the prototype using that of its model. The material system considered in the following analysis is graphite/epoxy with properties listed in Table 1.

Complete Similarity

The necessary condition for complete similarity between the model and its prototype is that all scaling laws are satisfied simultaneously. This requires that the scale factors of geometric parameters (length, radius, and thickness or number of plies) of both model and prototype be the same ($\lambda_R = \lambda_L = \lambda_t = \lambda_N$). Moreover, the model must have the same material properties as those of the prototype.

Table 1 Properties of AS4/3502 (Ref. 5)

Property	E_{11}	E_{22}	G_{12}	v_{12}
psi $\times 10^6$	19.85	1.43	0.82	0.293

The mode shapes of the model and the prototype must also be the same due to the condition of complete similarity. It turns out that for complete similarity $\lambda_{\bar{m}} = \lambda_{\bar{n}} = 1$, which leads to $\lambda_\xi = \lambda_n$ because $\lambda_L = \lambda_R$. Substituting these conditions into Eqs. (3–8), and after some manipulation, the following scaling laws are derived for the prediction of buckling pressure $\bar{N}_{yy}$:

$$\lambda_{\bar{N}_{yy}} = \lambda_R = \lambda_L = \lambda_t = \lambda_N \tag{9}$$

Equation (9) is a simple equation that can predict the critical lateral pressure of a prototype by using many models provided that geometry is completely scaled, the same materials are used for models and prototype, and ply level scaling[5] is used.

Partial Similarity

Complete similarity is very useful if there is freedom in designing scaled-down models. In many cases it is impossible to satisfy complete similarity between model and prototype due to the size, shape, material properties, and boundary and environmental conditions of the prototype. When at least one scaling law cannot be satisfied, distorted models with partial similarity are achieved. It is extremely useful to find a scaling law that can predict the behavior of the prototype with desired accuracy using a distorted model. In the following sections, two methodologies are presented in which partial similarity can be achieved. The first methodology is based on the derived scaling laws, and the second is based on a curve-fitting technique for constructing a surface for buckling pressure as a function of any two of the three parameters N, L, and R (number of plies, length, and radius, respectively)

Prediction Equations Using Scaling Laws

In determining which of the derived scaling laws yield the best prediction, the following assumptions for both model and prototype are taken into account: 1) $\lambda_{E_{ij}} = \lambda_{G_{ij}} = \lambda_{v_{ij}} = 1$ (same material), 2) cross-ply stacking sequences, and 3) same boundary conditions.

Each of Eqs. (3–8) is tested to predict the critical pressure of the prototype and the predicted results are compared with theoretical ones. Simitses et al.[4] showed that Eq. (4) yields the best accuracy for the case of distortion in the number of plies. Nevertheless, for the case of distortion in the cylinder radius, the study could not find a scaling law that can predict the critical pressure with accuracy because each scaling law is very sensitive to the distortion in radius. In this case, the models were incapable of predicting the critical pressure of prototype using only one of the scaling laws. The sixth and seventh columns of Table 2 present the theoretical critical lateral pressure $\bar{N}_{yy}$ and predicted results from Eq. (4), respectively. As the number of plies of the prototype increases, the discrepancy between predicted and theoretical results increases. Equation (4) is limited to a small range of distortions. Moreover, for Eq. (4) to yield the results in Table 2, the curvature parameter is kept the same for both model and prototypes as follows:

$$\lambda_z = 1 = \lambda_L^2 \big/ (\lambda_N\lambda_R) \Rightarrow \lambda_L = \sqrt{\lambda_N\lambda_R} \tag{10}$$

Table 2 Critical lateral pressure $\bar{N}_{yy}$(lb/in.) for partial similarity (distortion in N and L), model: $R = 7.0$ in., $L = 21.0$ in., $N = 20$, $t = 0.007$ in., $Z_m = Z_p = 450$, $Z = L^2/(R \times t)$

Config.	N	λ_N	R	λ_R	N_{yy}[a]	N_{yyp1}[b]	N_{yyp2}[c]	$e1$, %[d]	$e2$, %[e]	L	λ_L
$(0/90)_{5s}$	20	1	7	1	953.285	953.285	947.015	0	0.66	21.0	1
$(0/90)_{6s}$	24	1.2	7	1	1477.64	1372.73	1415.43	7.1	4.21	23.00	1.095
$(0/90)_{7s}$	28	1.4	7	1	1997.21	1868.44	1939.97	6.45	2.88	24.85	1.183
$(0/90)_{8s}$	32	1.6	7	1	2532.8	2440.41	2539.61	3.65	0.27	26.56	1.265
$(0/90)_{9s}$	36	1.8	7	1	3244.08	3088.64	3239.98	4.79	0.13	28.17	1.341
$(0/90)_{10s}$	40	2.0	7	1	4151.70	3813.14	4067.43	8.15	2.03	29.70	1.414
$(0/90)_{11s}$	44	2.2	7	1	5276.45	4613.89	5043.03	12.56	4.42	31.15	1.483
$(0/90)_{12s}$	48	2.4	7	1	6639.31	5490.92	6176.61	17.29	6.97	32.53	1.549
$(0/90)_{13s}$	52	2.6	7	1	8071.44	6444.20	7460.83	20.16	7.56	33.86	1.612

[a]Theoretical solutions.
[b]Predicted from Eq. (4).
[c]Predicted from Eq. (21).
[d]Discrepancy between theoretical and predicted critical pressures using Eq. (4).
[e]Discrepancy between theoretical and predicted critical pressures using Eq. (21).

In addition, the radius is assumed the same for model and prototype, which leads to an additional scaling factor as follows:

$$\lambda_L = \sqrt{\lambda_N} \tag{11}$$

To understand the reasons that Eq. (4) is the only scaling law that predicts the behavior of prototype for distortion in number of plies with good accuracy, let us express Eq. (4) in a different form. Using Eq. (4) and Eq. (8) and assuming that the number of half-waves in the axial direction are the same for both model and prototype, the resulting scaling law is as follows:

$$\lambda_{\bar{N}_{yy}} = \frac{\lambda_{\bar{D}_{12}}}{\lambda_L} = \frac{\lambda_{\bar{D}_{12}}}{(\lambda_N)^{0.5}} \tag{12}$$

Now let us expand the bending stiffness as a function of material properties, number of layers, and thickness as follows:

$$D_{ij} = \frac{1}{3}\sum^{N} \bar{Q}^k_{ij}\left(t_k^3 - t_{k-1}^3\right)$$

$$= \frac{t^3}{3}\sum^{N} \bar{Q}^k_{ij}\left[(1 - 3k + 3k^2) + \left(\frac{3}{2} - 3k\right)N + \frac{3}{4}N^2\right] \tag{13}$$

The scale factor for Eq. (13) can be represented by a polynomial as follows:

$$\lambda_{D_{ij}} = a_0 + a_1\lambda_N + a_2\lambda_N^2 \tag{14}$$

where a_0, a_1, and a_2 are functions of the number of layers for a specific material and stacking sequence. Equation (12) is the necessary scaling law that can be used to predict distortion in the number of layers with acceptable accuracy provided that Eq. (11) is also satisfied. Equation (12) is the scale factor $\lambda_{\bar{N}_{yy}}$, which is a polynomial function of λ_N [observing Eq. (13)]. In the next subsection it will be shown that $\lambda_{\bar{N}_{yy}}$ can be obtained as a polynomial function of λ_N by curve fitting the predicted results of the governing equation (1).

Prediction Equations Using Curve Fitting

Because the prediction depends on how sensitive the critical pressure is to the scaling law, it is sometimes impossible to find a scaling law that can predict results with acceptable accuracy for different distortions. When a scaling law is found, it is sometimes limited in its range of prediction. The limitation in range means that models that can predict the behavior of the prototype are a little smaller or little larger than the prototype and, therefore, there would be no advantage in testing these models.

Models satisfying all scaling laws will always give accurate prediction. Only if scaling requirements are violated, for instance, by neglecting a weak scaling law, would the prediction accuracy suffer.

The investigation of the scale factor $\lambda_{\bar{N}_{yy}}$ is necessary to establish inherent relationships among $\lambda_{\bar{N}_{yy}}$ and other scale factors. The buckling mode shapes are a function of the geometric parameters, boundary conditions, extensional stiffnesses A_{ij}, bending stiffnesses D_{ij}, and material properties. Therefore, the following equations are established:

$$\lambda_{\bar{m}}\lambda_{\bar{n}} = f(\lambda_L, \lambda_R, \lambda_N) \tag{15}$$

$$\lambda_{A_{ij}} = f(\lambda_N) \tag{16}$$

$$\lambda_{D_{ij}} = f(\lambda_N) \tag{17}$$

Substituting Eqs. (15–17) into Eqs. (3–8) and considering the combination of all obtained equations, the scale factor $\lambda_{\bar{N}_{yy}}$ has the form

$$\lambda_{\bar{N}_{yy}} = f(\lambda_N, 1/\lambda_R) \tag{18}$$

Furthermore Eq. (18) can be written as

$$\lambda_{\bar{N}_{yy}} = f_{\lambda_N}(\lambda_N) f_{\lambda_R}(1/\lambda_R) \tag{19}$$

The validity of combining the two functions (f_{λ_N}, f_{λ_R}) as a product is demonstrated in Ref. 6.

Observing Eq. (18) and the constant curvature assumption, any two of the scale factors λ_L, λ_R, and λ_N can be used in Eq. (19). Therefore, the product of two functions for the geometric scale factors would yield the buckling load (scale factor). The curve-fitting technique is used to construct a surface for the buckling load (scale factor) for stacking sequence $(0/90)_{ns}$, $n = 1, 2, 3, \ldots$, and distortion in R as follows:

$$\lambda_{\bar{N}_{yy}} = \left(-\frac{2.3643}{\lambda_R^4} + \frac{3.8959}{\lambda_R^3} - \frac{1.8703}{\lambda_R^2} + \frac{3.7304}{\lambda_R}\right) \times\left(-0.5533\lambda_N + 1.5927\lambda_N^2 - 1.0883\lambda_N^3 + 0.3898\lambda_N^4 - 0.0480\lambda_N^5\right) \tag{20}$$

For the sake of comparison to the prediction equation (4), let us assume that λ_R is equal to one for shells with stacking sequence $(0/90)_{ns}$, $n = 1, 2, 3, \ldots$. In this case we have distortion in N and L. Prediction equation (20) in this case is

$$\lambda_{\bar{N}_{yy}} = \left(-1.8766\lambda_N + 5.402\lambda_N^2 - 3.6912\lambda_N^3 + 1.3221\lambda_N^4 - 0.1628\lambda_N^5\right) \tag{21}$$

Results of prediction by Eq. (21) are compared to the ones obtained from Eq. (4) and presented in Table 2. The discrepancy between the theoretical and predicted values from Eq. (4) increases as the number of layers increase for the prototype. It is apparent that Eq. (21) has higher-order terms in the polynomial than Eq. (4) and, therefore, yields improved results. Although the prediction equations are developed from finite data points, theoretically they can predict the critical pressures of prototypes for any combination of λ_R and $\lambda_N (= \lambda_t)$ as long as R and N are greater than or equal to the model values.

Conclusion

The structural similitude theory is employed to establish scaling laws between model and prototype that are necessary to predict the behavior of the prototype. The investigation was performed on partial similarity (distortions in N, R, and L) and showed that similitude theory is limited in the prediction of prototype behavior. However, for complete similarity, a simple equation is obtained that can predict the critical pressure of the prototype using many models. Although this study only considers the distortions in the number of plies, length, and radius of laminated cylindrical shells, the method can be applied to find prediction equations with any other distortions such as boundary conditions, material properties, etc. The presented methodology can be used for other shell theories such as first-order shear deformation theory, which is more accurate for moderately thick shells.

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